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THE NUMBER OF NEAREST AND FARTHEST POINTS TO A COMPACTUM IN EUCLIDEAN SPACE

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ABSTRACT

A typical (in the sense of Baire category) compact A in $\mathbb{E}$, where $\mathbb{E}$ is either the Euclidean space $\mathbb{E}^s$, $s \geq 2$, or the separable Hilbert space $\mathbb{H}$, generates a dense subset $C^{n,m}(A)$ of the underlying space, such that every $x \in C^{n,m}(A)$ has exactly n nearest and m farthest points from A, whenever n and m are positive integers satisfying $n + m \leq \dim \mathbb{E} + 2$.

1. Introduction

Throughout, $\mathbb{E}$ is either a finite dimensional Euclidean space $\mathbb{E}^s$ of dimension $\dim \mathbb{E}^s = s \geq 2$, or a separable Hilbert space $\mathbb{H}$, both on the field of reals, with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. Every nonempty set $A \subset \mathbb{E}$ generates a set-valued mapping

$$P(e, A) = \{y \in A \colon |e - y| = d(e, A)\}$$

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called **metric projection** or **nearest point mapping**, with a distance function $d(e,A) = \inf\{|e\ z|\colon z \in A\}$, defined on $\mathbb{E}$. Each $y \in P(e,A)$ is called a point (or element) of best approximation of A from $e \in \mathbb{E}$.

Likewise, one defines the **metric antiprojection** or the **farthest point mapping**

$$Q(e,A) = \{y \in A\colon |e - y| = f(e,A)\},$$

with the farthest distance function$f(e,A) = \sup\{|e - z|\colon z \in A\}$, provided A is nonempty and bounded.

Denote by $\mathcal{K}$ the complete metric space of all nonempty compact subsets of $\mathbb{E}$ endowed with the Pompeiu-Hausdorff metric χ. The cardinal number of a set A will be $\operatorname{card} A$. As usual $\mathbb{N}$ stands for the set of all natural numbers.

Let $X \in \mathcal{K}$ and $n, m \in \mathbb{N}$. The sets

$$L_P^n(X) = \{e \in \mathbb{E}\colon \operatorname{card} P(e,X) = n\} \tag{1.1}$$

and

$$\hat{L}_P^n(X) = \{e \in \mathbb{E}\colon \operatorname{card} P(e,X) \geq n\} \tag{1.2}$$

are called respectively, n-**locus** and **upper** n-**locus** of the metric projection generated by X. Similarly, if in (1.1) and (1.2), $P(e,X)$ is replaced by $Q(e,X)$, and n by m, one obtains the definitions of the m-**locus** $L_Q^m(X)$ and the **upper** m-**locus** $\hat{L}_Q^m(X)$ of the metric antiprojection generated by X. Further, the sets

$$C^{n,m}(X) = L_P^n(X) \cap L_Q^m(X) \quad \text{and} \quad \hat{C}^{n,m}(X) = \hat{L}_P^n(X) \cap \hat{L}_Q^m(X)$$

are called respectively **common** (n,m)-**locus** and **upper common** (n,m)-**locus**, generated by X. Obviously, for any set $X \in \mathcal{K}$

$$\hat{C}^{n,m}(X) \supset \bigcup_{i \geq n} \bigcup_{j \geq m} C^{i,j}(X).$$

A set A in a complete metric space M is called residual in M if its complement $M \backslash A$ is of first Baire category in M, i.e., if A contains a dense G_δ subset of M. If A is residual in M, any element $a \in A$ is called "roughly speaking" a **typical** element of M.

By classical results established in a more general framework by Stechkin [St], Edelstein [Ed] and Asplund [As], it is known that for every $X \in \mathcal{K}$ the sets $L_P^1(X)$ and $L_Q^1(X)$ are residual in $\mathbb{E}$. More recently Zamfirescu [Za] has proved that for a typical $X \in \mathcal{K}$ (i.e., for every X of a residual subset of $\mathcal{K}$) the set $\mathbb{E}^s \backslash L_P^1(X)$ is dense in $\mathbb{E}^s$. De Blasi and Myjak [BM] extended Zamfirescu's result to infinite dimensional spaces. For the antiprojection a similar property is also true [Zh1]. Thus for a typical $X \in \mathcal{K}$ both $\mathbb{E}\backslash L_P^1(X)$ and $\mathbb{E}\backslash L_Q^1(X)$ are dense in $\mathbb{E}$. However, each of the latter sets is of first Baire category and so their intersection might fail to be dense. In this paper it will be shown that this is not the case, and we prove even more.

In fact, if $n, m \in \mathbb{N}$ are arbitrary then a typical $X \in \mathcal{K}$ has upper common (n,m)-locus $\hat{C}^{n,m}(X)$ which is dense in $\mathbb{E}$ (Proposition 3.1). Making use of this, a stronger result is established in both the finite-dimensional case, and the Hilbert space case, namely, for a typical $X \in \mathcal{K}$ also the common locus $C^{n,m}(X)$ is dense in $\mathbb{E}$ (Theorem 5.3), and this is the first known example of a compactum with either property.

Also, it turns out for the Euclidean space $\mathbb{E}^s$, $s \geq 2$, that the common (n,m)-locus of a typical $X \in \mathcal{K}$ is empty whenever $n + m > s + 2$; thus the sequence of common loci of X produces a finite partition of the space $\mathbb{E}^s$ on dense sets. Intersections of common loci of typical compacta varying in the whole space with affine subspaces of $\mathbb{E}$ are also investigated.

For metric projections and antiprojections, the corresponding loci $L_P^n(X)$ and $L_Q^m(X)$ have been studied by de Blasi [Bl] and de Blasi and Zamfirescu [BZ]. A comprehensive discussion of the properties of the maps $P(\cdot, X)$ and $Q(\cdot, X)$, and additional bibliography, can be found in Singer [Si1], [Si2], and Dontchev and Zolezzi [DZ]. For further properties about typical compacta, or closed bounded sets, and the number of nearest or farthest points, see Motzkin, Straus and Valentine [MSV], Kuz'minykh [Ku], Radul [Ra], Zhivkov [Zh2].

The following theorem, due to Brouwer [Br] and Miranda [Mi] (see also [RM], p. 171), is used in the sequel:

Brouwer-Miranda Theorem. *Let $I^t \subset \mathbb{E}^s$ be a bounded polyhedron of the form $\{x \in \mathbb{E}^s\colon \mid < v_i, x - a > \mid \leq t, i = 1, \ldots, s\}$ where $a \in \mathbb{E}^s$, $t > 0$, and $v_1, \ldots, v_s$ are linearly independent vectors. For $i = 1, \ldots, s$, let $L_i^{\pm}(t) = \{x \in I^t\colon < v_i, x - a >= \pm t\}$ and $g_i\colon I^t \to \mathbb{R}$ be continuous functions such that $g_i(x) < 0$ if $x \in L_i^-(t)$, $g_i(x) > 0$ if $x \in L_i^+(t)$. Then there exists a point $\hat{x} \in I^t$ such that $g_i(\hat{x}) = 0$ for all $i = 1, \ldots, s$.*

2. Notation

A point-to-set mapping $F\colon Z \to Y$ with nonempty images, where Z and Y are topological spaces, is upper semi-continuous at $z_0 \in Z$ provided for every open set $U \supset F(z_0)$ there is an open set $V \subset Z$, $z_0 \in V$, such that $U \supset F(z)$ whenever

$z \in V$.

In $\mathcal{K}$, open and closed balls with center $X \in \mathcal{K}$ and radius r are denoted by $\mathcal{B}(X,r)$ and $\mathcal{B}[X,r]$, respectively. In $\mathbb{E}$, open and closed balls with center $x \in \mathbb{E}$ are written as $\mathsf{B}(x,r)$ and $\mathsf{B}[x,r]$. Further, $\mathsf{S}(x,r)$ stands for the sphere with the same center and radius, and S is the unit sphere, i.e., $\mathsf{S} = \mathsf{S}(\theta,1)$, where θ is the origin of $\mathbb{E}$. B is the closed unit ball.

An affine subspace is a translate of a linear subspace (not necessarily closed) of $\mathbb{E}$, and a hyperplane is a closed affine subspace of codimension one in $\mathbb{E}$.

For a subset X of $\mathbb{E}$, $\operatorname{co} X$ stands for the convex hull of X. Given $X \subset \mathbb{E}$, denote by $\operatorname{span} X$ the minimal subspace of $\mathbb{E}$ which contains X, and by $\operatorname{lin} X$, denote its linear hull, i.e., the minimal affine subspace containing X. Obviously, $\operatorname{lin}(\{\theta\} \cup X) = \operatorname{span} X$. For any finite set $X = \{x_1, \dots, x_k\} \subset \mathbb{E}$, $k \geq 2$, define

$$\operatorname{sep}(x_1, \dots, x_k) = \min\{|x_i - x_j| : i \neq j,\ i,j = 1, \dots, k\}.$$

If X is finite, $X = \{x_1, \dots, x_k\}$, we make use of either X or $(x_1, \dots, x_k)$ as argument of any of the operators: co, lin, span, sep etc.

Any k points $x_1, \dots, x_k \in \mathbb{E}$ are said to be in general position provided $\dim \operatorname{lin}(x_1, \dots, x_k) = k - 1$. Certainly this is so if, and only if, the vectors $x_2 - x_1, \dots, x_k - x_1$ are linearly independent.

Put $[a_1, \dots, a_k] = \operatorname{co}(a_1, \dots, a_k)$ for $a_i \in \mathbb{E}$, $i = 1, \dots, k$. This set is a $(k-1)$-dimensional simplex whenever its vertices $a_1, \dots, a_k$ are in general position. A facet (resp. m-dimensional facet) of a simplex A is the convex hull of a subset (resp. subset of $m+1$ elements in general position) of its vertices. The 0-dimensional facets are the vertices themselves.

For $v \in \mathbb{E}$, as usual, $\ker v$ is the kernel of v (v is viewed as a bounded functional), i.e., $\ker v = \{x \in \mathbb{E} : \langle v, x \rangle = 0\}$.

3. Compacta with dense upper common (n,m)-locus

PROPOSITION 3.1: *Let H be a closed subspace of $\mathbb{E}$ and let $n,m \in \mathbb{N}$, $n+m-2 \leq \dim H$. Then for a typical $X \in \mathcal{K}$ the set $\hat{C}^{n,m}(X) \cap H$ is dense in H.*

Proof: The case $\dim H = 0$ is trivial. Let $\dim H \geq 1$. For $h \in H$ and $r > 0$ set

$$\mathcal{M}^{n,m}_{h,r} = \{X \in \mathcal{K} : \hat{C}^{n,m}(X) \cap H \cap \mathsf{B}(h,r) \neq \emptyset\}$$

and

$$\hat{\mathcal{K}}^{n,m} = \{X \in \mathcal{K} : \hat{C}^{n,m}(X) \cap H \text{ is dense in } H\}.$$

Suppose D is a countable dense subset of H. Evidently,

$$\bigcap_{h\in D}\bigcap_{r^{-1}\in\mathbb{N}}\mathcal{M}^{n,m}_{h,r}\subset\hat{\mathcal{K}}^{n,m}. \tag{3.1}$$

To complete the proof it suffices to show that each set $\mathcal{M}^{n,m}_{h,r}$ on the left side of (3.1) contains an open dense subset of $\mathcal{K}$. This is a consequence of the following

Lemma 3.2: *Given a compact $A_0\in\mathcal{K}$, a closed subspace $H\subset\mathbb{E}$, $\dim H\geq 1$, a point $h_0\in H$, positive numbers λ, $r>0$ and integers $n,m\in\mathbb{N}$, $n+m-2\leq\dim H$, there exist $A_1\in\mathcal{K}$ and $\mu>0$ with $\mathcal{B}(A_1,\mu)\subset\mathcal{B}(A_0,\lambda)$, such that for every $X\in\mathcal{B}(A_1,\mu)$ one has*

$$\hat{C}^{n,m}(X)\cap H\cap \mathsf{B}(h_0,r)\neq\emptyset.$$

Proof: Without loss of generality assume $h_0\notin A_0$. Otherwise, take $A'\in\mathcal{B}(A_0,\lambda/2)$ such that $h_0\notin A'$ and prove the lemma for A' and $\lambda'=\lambda/2$ with h_0 and r unchanged. Suppose also $h_0=\theta$, i.e., all the coordinates of h_0 are equal to 0 in a new coordinate system obtained after translation by $-h_0$.

Take $\bar{a}\in P(\theta,A_0)$, $\bar{b}\in Q(\theta,A_0)$. Let α_0 and β_0, $0<\alpha_0<1$, $\beta_0>1$, be chosen so that the points $a_0=\alpha_0\bar{a}$ and $b_0=\beta_0\bar{b}$ satisfy

$$d(a_0,A_0)<\lambda/2,\quad d(b_0,A_0)<\lambda/2.$$

Denote by H^* the orthogonal complement of H, i.e.,

$$H^*=\{y\in\mathbb{E}\colon <y,x>=0,\forall x\in H\},$$

and by $\pi\colon\mathbb{E}\to H$ the orthogonal projection operator onto H (along H^*). Our next goal is to find

$$a_i\in|a_0|\mathsf{S},\ d(a_i,A_0)<\lambda/2,\quad\text{for } i=1,\ldots,n-1, \tag{3.2}$$

$$b_j\in|b_0|\mathsf{S},\ d(b_j,A_0)<\lambda/2,\quad\text{for } j=1,\ldots,m-1, \tag{3.3}$$

such that the vector system

$$V=\{a_1-a_0,\ldots,a_{n-1}-a_0,b_1-b_0,\ldots,b_{m-1}-b_0\}, \tag{3.4}$$

regarded as a set of functionals acting on H, be linearly independent, i.e., the set πV be linearly independent. If $n=1$ or $m=1$, V is reduced to $\{b_j-b_0\colon j=1,\ldots,m-1\}$ or $\{a_i-a_0\colon i=1,\ldots,n-1\}$ correspondingly, and in case $n=m=1$ the proof is simplified.

Proceed by induction: As H^* is a closed proper subspace of $\mathbb{E}$ and $a_0 + H^*$ is nowhere dense in $|a_0|\mathsf{S}$, there is $a_1 \in |a_0|\mathsf{S}\backslash(a_0 + H^*)$ satisfying (3.2) such that $\pi(a_1 - a_0)$ is non-zero on H. If $n = 1$ we start with finding b_1.

Assume by induction that $a_1, \ldots, a_{k-1}$ for $k < n$ satisfy (3.2), and $\pi(a_i - a_0)$ for $i = 1, \ldots, k-1$ are linearly independent. Certainly, $k - 1 < \dim H$ since $m \geq 1$. So, for $H_{k-1} = \bigcap_{i=1}^{k-1} \ker(a_i - a_0) \cap H$, its orthogonal complement H_{k-1}^* in $\mathbb{E}$ has positive codimension, i.e., $H_{k-1}^* \neq \mathbb{E}$. Then $a_0 + H_{k-1}^*$ is nowhere dense in $|a_0|\mathsf{S}$ and there is $a_k \in |a_0|\mathsf{S}\backslash(a_0 + H_{k-1}^*)$ satisfying (3.2). Since H_{k-1}^* contains both $a_i - a_0$ and $\pi(a_i - a_0)$ for $i = 1, \ldots, k-1$ then $\pi(a_k - a_0)$ is independent to the previously projected vectors.

After finding a_{n-1}, find b_1 in a similar way if $m > 1$. In this case $n - 1 < \dim H$, and the set $b_0 + H_{n-1}^*$ does not cover the sphere $|b_0|\mathsf{S}$, where H_{n-1}^* is defined as above. Hence there is b_1 satisfying (3.3) such that $b_1 - b_0 \notin H_{n-1}^*$, which entails the linear independence of the latter vector with all $a_i - a_0$, $i = 1, \ldots, n-1$. We proceed by induction until our goal is reached.

Since $\bar{a}$ is the nearest point to a_0 in A_0, and $\bar{b}$ is also the nearest point to b_0 in A_0, it is clear that for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$

$$d(a_i, A_0) \geq d(a_0, A_0), \ d(b_j, A_0) \geq d(b_0, A_0). \tag{3.5}$$

Let

$$A_1 = A_0 \cup \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{m-1}\}, \quad G = \operatorname{span}(\pi V)$$

and $\eta > 0$ be chosen so that

$$4\eta < \min\{r, \operatorname{sep}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}), d(a_0, A_0), d(b_0, A_0)\}. \tag{3.6}$$

Furthermore, for $t > 0$ denote by I^t the "distorted" cube

$$G \cap \{x \in H \colon |< a_i - a_0, x >| \leq t, \ i = 1, \ldots, n-1, \\ |< b_j - b_0, x >| \leq t, \ j = 1, \ldots, m-1\}$$

and put

$$L_i^{\pm}(t) = \{x \in I^t \colon < a_i - a_0, x >= \pm t\}, \quad i = 1, \ldots, n-1,$$
$$M_j^{\pm}(t) = \{x \in I^t \colon < b_j - b_0, x >= \pm t\}, \quad j = 1, \ldots, m-1.$$

Since the vectors in (3.4) are linearly independent, I^t is a bounded polyhedron containing $\theta = h_0$ in its relative interior with respect to G and its diameter vanishes as $t \to 0$. Now fix $t > 0$ such that

$$I^t \subset \mathsf{B}(\theta, \eta). \tag{3.7}$$

For arbitrary $X \in \mathcal{B}[A_1,\eta]$ and $i = 0,\dots,n-1$, $j = 0,\dots,m-1$, set

$$X(a_i) = X \cap \mathsf{B}[a_i,\eta],\ X(b_j) = X \cap \mathsf{B}[b_j,\eta],\ \tilde{X} = X \cap (A_1 + \eta\mathsf{B}). \tag{3.8}$$

Since $\chi(X, A_1) \le \eta$ and η verifies (3.6), one can easily show, having in mind (3.5), that the above sets are nonempty, pairwise disjoint, compact and

$$\Big(\bigcup_{i=0}^{n-1} X(a_i)\Big) \bigcup \Big(\bigcup_{j=0}^{m-1} X(b_j)\Big) \bigcup \tilde{X} = X.$$

In the next step of the proof one has to find μ,

$$0 < \mu < \min\{\eta, \lambda/2\}, \tag{3.9}$$

such that for every $X \in \mathcal{B}(A_1,\mu)$

$$d(h, X(a_0)) < d(h, X(a_i)) \quad \text{if } h \in L_i^-(t), \tag{3.10}$$

$$d(h, X(a_0)) > d(h, X(a_i)) \quad \text{if } h \in L_i^+(t), \tag{3.11}$$

for $i = 1,\dots,n-1$, and

$$f(h, X(b_0)) < f(h, X(b_j)) \quad \text{if } h \in M_j^-(t), \tag{3.12}$$

$$f(h, X(b_0)) > f(h, X(b_j)) \quad \text{if } h \in M_j^+(t), \tag{3.13}$$

for $j = 1,\dots,m-1$.

In order to prove the existence of such μ, let $X = A_1$. Then for $h \in M_j^{\pm}(t)$, $j = 1,\dots,m-1$, one has

$$\begin{aligned} f^2(h, A_1(b_0)) - f^2(h, A_1(b_j)) &= |b_0 - h|^2 - |b_j - h|^2 \\ &= 2\langle b_j - b_0, h\rangle = \pm 2t. \end{aligned}$$

Similarly, for $h \in L_i^{\pm}(t)$, $i = 1,\dots,n-1$, it can be shown that

$$d^2(h, A_1(a_0)) - d^2(h, A_1(a_i)) = \pm 2t.$$

For $i = 0,\dots,n-1$, $j = 0,\dots,m-1$, the mappings $X \to X(a_i)$ and $X \to X(b_j)$ are continuous, the sets $L_i^{\pm}(t)$, $M_j^{\pm}(t)$ are compact and (3.10)–(3.13) hold true when $X = A_1$, whence there is μ verifying (3.9) such that for every $X \in \mathcal{B}(A_1,\mu)$ (3.10)–(3.13) are satisfied.

Now to prove the statement of the lemma with A_1 and μ obtained above, observe first that $\mathcal{B}(A_1,\mu) \subset \mathcal{B}(A_0,\lambda)$ since $\chi(A_0,A_1) < \lambda/2$ and $\mu < \lambda/2$. Suppose $X \in \mathcal{B}(A_1,\mu)$ is arbitrary. It remains to show that

$$\hat{C}^{n,m}(X) \cap H \cap \mathsf{B}(\theta,r) \neq \emptyset.$$

Apply the Brouwer-Miranda theorem to the functions

$$\begin{aligned} d(h,X(a_0)) - d(h,X(a_i)),\ i=1,\dots,n-1,\\ f(h,X(b_0)) - f(h,X(b_j)),\ j=1,\dots,m-1, \end{aligned}$$

all defined on I^t, and obtain a point $\hat{h} \in I^t$ at which all of them vanish, i.e.,

$$d(\hat{h},X(a_0)) = d(\hat{h},X(a_i)),\ i=1,\dots,n-1, \tag{3.14}$$

$$f(\hat{h},X(b_0)) = f(\hat{h},X(b_j)),\ j=1,\dots,m-1. \tag{3.15}$$

We claim that $\hat{h} \in \hat{C}^{n,m}(X)$. In view of (3.7) and (3.8) one has

$$\begin{gathered} d(\hat{h},X(a_i)) \le |\hat{h}| + d(\theta,X(a_i)) < |a_0| + 2\eta, \quad i=0,\dots,n-1,\\ d(\hat{h},X(b_j)) \ge -|\hat{h}| + d(\theta,X(b_j)) > |b_0| - 2\eta, \quad j=0,\dots,m-1,\\ d(\hat{h},\tilde{X}) \ge -|\hat{h}| + d(\theta,\tilde{X}) > |\bar{a}| - 2\eta. \end{gathered}$$

As $4\eta < d(a_0,A_0) = |\bar{a}| - |a_0| < |b_0| - |a_0|$, the above inequalities imply

$$P(\hat{h},X) \subset \bigcup_{i=0}^{n-1} X(a_i)$$

and consequently in view of (3.14) it follows that $d(\hat{h},X) = d(\hat{h},X(a_i))$, i.e.,

$$P(\hat{h},X) \cap X(a_i) \neq \emptyset \quad \text{for } i=0,\dots,n-1.$$

Similarly,

$$\begin{gathered} f(\hat{h},X(b_j)) \ge f(\theta,X(b_j)) - |\hat{h}| > |b_0| - 2\eta, \quad j=0,\dots,m-1,\\ f(\hat{h},X(a_i)) \le f(\theta,X(a_i)) + |\hat{h}| < |a_0| + 2\eta, \quad i=0,\dots,n-1,\\ f(\hat{h},\tilde{X}) \le f(\theta,\tilde{X}) + |\hat{h}| < |\bar{b}| + 2\eta. \end{gathered}$$

As $4\eta < d(b_0,A_0) = |b_0| - |\bar{b}| < |b_0| - |a_0|$, then

$$Q(\hat{h},X) \subset \bigcup_{j=0}^{m-1} X(b_j)$$

whence, by (3.15), $f(\hat{h}, X) = f(\hat{h}, X(b_j))$, i.e.,

$$Q(\hat{h}, X) \cap X(b_j) \neq \emptyset \quad \text{for } j = 0, \dots, m-1.$$

The sets $X(a_i)$, $X(b_j)$, for $i = 0, \dots, n-1$, $j = 0, \dots, m-1$, are pairwise disjoint which implies

$$\operatorname{card} P(\hat{h}, X) \geq n \quad \text{and} \quad \operatorname{card} Q(\hat{h}, X) \geq m.$$

Moreover, $\hat{h} \in I^t \subset G \cap \mathsf{B}(h_0, r)$. Therefore $\hat{h} \in \hat{C}^{n,m}(X) \cap H \cap \mathsf{B}(h_0, r)$ whenever $X \in \mathcal{B}(A_1, \mu)$. The proof is completed. ∎

Remark 3.3: Proposition 3.1 remains valid if H is an arbitrary affine subspace of $\mathbb{E}$. This follows by category arguments, after approximating H by a dense sequence of finite-dimensional affine subspaces contained in H.

4. Geometrical lemma

Suppose $A = [a_1, \dots, a_{s+1}]$ is a s-dimensional simplex in $\mathbb{E}^s$. Associate with it the set $\Lambda(A)$, denoted also by $\Lambda(a_1, \dots, a_{s+1})$, of all $e \in \mathbb{E}^s$ such that:

$$e = \sum_{i=1}^{s+1} \lambda_i a_i, \quad \sum_{i=1}^{s+1} \lambda_i = 1, \quad \lambda_i \in \mathbb{R},$$

$$\text{there exist } \lambda_{i_1}, \dots, \lambda_{i_k},\ 1 \leq k \leq s, \text{ such that } \sum_{j=1}^{k} \lambda_{i_j} = 0.$$

The set $\Lambda(A)$ contains all points in $\mathbb{E}^s$ which allow that partial sums of their baricentric coordinates, with respect to A, be equal to 0. Actually, $\Lambda(A)$ is the union of all hyperplanes containing facets (of various dimensions) of A and parallel to the linear hulls of the remaining vertices. As $\Lambda(A)$ is a finite union of (in fact $2^{s+1} - 2$) hyperplanes, it is nowhere dense in $\mathbb{E}^s$.

LEMMA 4.1: *Let $A = [a_1, \dots, a_{s+1}] \subset \mathbb{E}^s$ be a non-degenerate simplex and $a_0 \in \mathbb{E}^s \setminus \Lambda(A)$. Then for every j, $1 \leq j \leq s+1$, the simplex*

$$A_j = [a_0, a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{s+1}]$$

is nondegenerate and $a_0 \notin \Lambda(A)$ if, and only if, $a_j \notin \Lambda(A_j)$.

Proof: The statement follows by routine calculations, after representing a_0 in barycentric coordinates with respect to A. ∎

LEMMA 4.2: *Let A be a non-degenerate simplex in $\mathbb{E}^s$ and $a_0 \notin \Lambda(A)$. Suppose the set of the vertices of A is arbitrarily split in two sets $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_m\}$ such that $n \geq 2$, $n + m = s + 1$. Then the vector system*

$$b_2 - b_1, \ldots, b_n - b_1, c_1 - a_0, \ldots, c_m - a_0 \tag{4.1}$$

is linearly independent (in fact it is a basis of $\mathbb{E}^s$).

Proof: If $m = 1$ then the definition of $\Lambda(A)$ implies (4.1), else denote

$$U = \{u_i = b_{i+1} - b_1 \colon i = 1, \ldots, n-1\}, \tag{4.2}$$

$$V = \{v_j = c_{j+1} - c_1 \colon j = 1, \ldots, m-1\}. \tag{4.3}$$

The vectors in (4.2) and (4.3) are independent and so is their union.

Set $H = c_1 + \operatorname{span}(U \cup V)$. The hyperplane H is contained in $\Lambda(A)$. Indeed, if $h \in H$ then for some $\mu_i \in \mathbb{R}$, $i = 1, \ldots, n-1$, and $\nu_j \in \mathbb{R}$, $j = 1, \ldots, m-1$, we have

$$\begin{aligned} h =& c_1 + \sum_{i=1}^{n-1} \mu_i u_i + \sum_{j=1}^{m-1} \nu_j v_j \\ =& -(\mu_1 + \cdots + \mu_{n-1}) b_1 + \mu_1 b_2 \cdots + \mu_{n-1} b_n \\ &+ (1 - \nu_1 - \cdots - \nu_{m-1}) c_1 + \nu_1 c_2 + \cdots + \nu_{m-1} c_m. \end{aligned}$$

Giving h its barycentric coordinates shows that a partial sum of them (first n) is zero, so $h \in \Lambda(X)$. Now, since $a_0 \notin H$, the vectors

$$u_1, \ldots, u_{n-1}, v_1, \ldots, v_{m-1}, c_1 - a_0 \tag{4.4}$$

are linearly independent. In order to verify the statement of the lemma suppose

$$\sum_{i=1}^{n-1} \tau_i (b_{i+1} - b_1) + \sum_{i=n}^{s} \tau_i (c_{i-n+1} - a_0) = 0$$

for $\tau_i \in \mathbb{R}$, $i = 1, \ldots, s$, whence

$$\tau_1 u_1 + \cdots + \tau_{n-1} u_{n-1} + \tau_{n+1} v_1 + \cdots + \tau_s v_{m-1} + (\tau_n + \cdots + \tau_s)(c_1 - a_0) = 0,$$

which, in view of the linear independence of the vectors in (4.4), implies $\tau_i = 0$ for $i = 1, \ldots, s$. Therefore the vectors (4.1) are linearly independent. The proof is completed. ∎

For a finite set A and an integer $i \geq 2$, we define $\mathcal{E}^i(A)$ to be the set of all couples $\{B, C\}$ of nonempty subsets of A such that $B \cap C = \emptyset$ and $\operatorname{card} B + \operatorname{card} C = i$.

The next geometrical lemma actually shows that among the finite sets there are "quite many" of them with empty common (n, m)-locus on subspaces, whenever $n + m$ is large enough.

LEMMA 4.3: *Given a nonempty finite set $X \subset \mathbb{E}^s$, a subspace $H \subset \mathbb{E}^s$, $\dim H = k$, and a number $\varepsilon > 0$, there exists a finite set A, $\operatorname{card} A \geq k+2$, such that: $\chi(A, X) < \varepsilon$ and for every $\{B, C\} \in \mathcal{E}^{k+2}(A)$ there exist unique $h \in H$, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ satisfying*

$$B = A \cap \mathsf{S}(h, r_1), \quad C = A \cap \mathsf{S}(h, r_2). \tag{4.5}$$

Proof: Assume $k \geq 1$; as in case $k = 0$ one can take a finite set A with $\operatorname{card} A \geq 2$ and $\chi(X, A) < \varepsilon$ such that $\operatorname{card}(A \cap t\mathsf{S}) < 2$ whenever $t > 0$. Let H^* stand for the orthogonal complement of H and let $N = \{\mathsf{n}_1, \ldots, \mathsf{n}_{s-k}\}$ be a basis of H^*. If $k = s$ then $H^* = \{\theta\}$ and $N = \emptyset$. Suppose $X = \{x_1, \ldots, x_l\}$ with $l \geq k+2$, which is no loss of generality.

Construct A by induction. The first $k+1$ points $\{a_1, \ldots, a_{k+1}\} = A_1$ are taken to satisfy $|x_i - a_i| < \varepsilon$, $i = 1, \ldots, k+1$, and the points from $(a_1 + N) \cup A_1$ to be in general position.

Since $\mathsf{n}_1, \ldots, \mathsf{n}_{s-k}, a_2 - a_1, \ldots, a_{k+1} - a_1$ are linearly independent, the subspace $G = \bigcap_{i=1}^{k} \ker(a_{i+1} - a_1)$ is complementary to H. Hence the affine subspace of all centers of spheres containing A_1, being parallel to G, intersects H at a single point h_0. Thus $A_1 \subset \mathsf{S}(h_0, r_0)$ for some $r_0 > 0$.

Next take a_{k+2} such that $|x_{k+2} - a_{k+2}| < \varepsilon$, and

$$a_{k+2} \notin \Big(\bigcup_{i=1}^{k+1} \Lambda((a_i + N) \cup A_1) \Big) \bigcup \mathsf{S}(h_0, r_0), \tag{4.6}$$

which is possible as the union of sets in (4.6) is nowhere dense in $\mathbb{E}^s$.

To verify the statement of the lemma at this step, let $\{B, C\} \in \mathcal{E}^{k+2}(A_2)$, where

$$A_2 = A_1 \cup \{a_{k+2}\}, \ \mathsf{B} = \{b_1, \ldots, b_n\}, \ C = \{c_1, \ldots, c_m\}, \ n + m = k+2.$$

If either $\operatorname{card} B = 1$ or $\operatorname{card} C = 1$, the statement is deduced from (4.6) and Lemma 4.1 similarly to the way of finding h_0. Otherwise, note at first that it is possible to change the names of B and C, and to reorder their elements if necessary, so that $b_1 = a_1$ and $c_m = a_{k+2}$, since we have to ascertain only that there are $b \in B$ and $c \in C$ satisfying $c \notin \Lambda((b+N) \cup (A_2 \backslash \{c\}))$.

Further, the vectors

$$\mathsf{n}_1, \ldots, \mathsf{n}_{s-k}, b_2 - b_1, \ldots, b_n - b_1, c_1 - c_m, \ldots, c_{m-1} - c_m \tag{4.7}$$

are linearly independent as a consequence of Lemma 4.2. Therefore, the subspaces

$$G_1 = \bigcap_{i=1}^{n-1} \ker(b_{i+1} - b_1), \quad G_2 = \bigcap_{j=1}^{m-1} \ker(c_j - c_m)$$

are independent to $\bigcap_{i=1}^{s-k} \ker \mathsf{n}_i = H$ and so is their intersection. However, $\dim(G_1 \cap G_2) = s - k$, whence H and $G_1 \cap G_2$ are complementary subspaces in $\mathbb{E}^s$ and they meet only at θ. Since the centers of the spheres containing B and C are parallel to G_1 and G_2 respectively, there exist uniquely determined $h \in H$, $r_1 > 0$ and $r_2 > 0$ satisfying (4.5) with A_2 instead of A. To check $r_1 \neq r_2$ observe that, due to (4.6), A_2 is not contained in any sphere centered at a point in H. Note that in the process of constructing the set A_2 we also order its elements.

Suppose now by induction that for $j \in \mathbb{N}$, $2 \leq j < l - k$ there is an ordered set $A_j = \{a_1, \ldots, a_{k+j}\}$ such that the following are true:

$$|x_i - a_i| < \varepsilon, \quad i = 1, \ldots, k+j;$$

for every $\{B, C\} \in \mathcal{E}^{k+2}(A_j)$ there exist unique $h \in H$, $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$ satisfying

$$B = A_j \cap \mathsf{S}(h, r_1), \quad C = A_j \cap \mathsf{S}(h, r_2). \tag{4.8}$$

Note that the induction assumption implies that no $k+2$ points in A_j belong to a sphere centered in H and no $k+3$ points from A_j are in the union of two concentric spheres with centers in H.

For every $\{B, C\} \in \mathcal{E}^{k+2}(A_j)$, denote by $U(B, C)$ the union of the two concentric spheres which are uniquely determined by (4.8). Now take a_{k+j+1},

$$|x_{k+j+1} - a_{k+j+1}| < \varepsilon, \tag{4.9}$$

such that with $A_{j+1} = A_j \cup \{a_{k+j+1}\}$ the following are true:

$$a_{k+j+1} \notin \bigcup \{U(B, C) \colon \{B, C\} \in \mathcal{E}^{k+2}(A_j)\}; \tag{4.10}$$

for every $A' \subset A_j$, $\operatorname{card} A' = k+1$,

$$a_{k+j+1} \notin \bigcup \{\Lambda((a+N) \cup A') \colon a \in A'\}. \tag{4.11}$$

Since the sets in (4.10) and (4.11) are nowhere dense in $\mathbb{E}^s$, it is always possible to find a_{j+1} satisfying (4.9) under the conditions imposed by (4.10) and (4.11).

As in the case $j = 1$, it can be proved that for $\{B, C\} \in \mathcal{E}^{k+2}(A_{j+1})$ there are unique $h \in H$, $r_1 > 0$ and $r_2 > 0$ such that

$$B \subset A_{j+1} \cap \mathsf{S}(h, r_1), \quad C \subset A_{j+1} \cap \mathsf{S}(h, r_2). \tag{4.12}$$

To observe that $r_1 \neq r_2$ and that the inclusions in (4.12) are actually replaced by equalities, make use of (4.10). ∎

COROLLARY 4.4: *For any subspace H of $\mathbb{E}^s$, there exists a dense collection $\mathcal{A}_H$ in $\mathcal{K}$ of finite sets such that for every $A \in \mathcal{A}_H$ and $h \in H$ one has*

$$\operatorname{card}(P(h, A) \cup Q(h, A)) \leq \dim H + 2.$$

Proof: As the set of the finite subsets of $\mathbb{E}^s$ is dense in $\mathcal{K}$, there is a dense set $\mathcal{A}_H \subset \mathcal{K}$ obtained by virtue of Lemma 4.3 so that for $A \in \mathcal{A}_H$: No $k + 3$ points in A belong to a union of two concentric spheres centered in H. ∎

5. Compacta with dense common (n, m)-locus

PROPOSITION 5.1: *Let H, $\dim H = k$, be a finite-dimensional subspace of $\mathbb{E}$, and let $n, m \in \mathbb{N}$ satisfy $n + m = k + 2$. Then for a typical $X \in \mathcal{K}$ the set $C^{n,m}(X) \cap H$ is dense in H.*

Proof: Fix $h_0 \in H$. For $R > 0$ and $\varepsilon > 0$ denote by $\mathcal{M}_{R,\varepsilon}$ the set of all $X \in \mathcal{K}$ with the property: If $h \in \mathsf{B}[h_0, R] \cap H$ and $\operatorname{card} P(h, X) + \operatorname{card} Q(h, X) > k + 2$, then for every set $W \subset P(h, X) \cup Q(h, X)$ with $\operatorname{card} W = k+3$, one has $\operatorname{sep} W < \varepsilon$.

CLAIM: *$\mathcal{M}_{R,\varepsilon}$ contains an open and dense subset of $\mathcal{K}$.*

Let $A_0 \in \mathcal{K}$ and $\lambda > 0$ be arbitrary. It suffices to prove that there exists $A_1 \in \mathcal{K}$ and $\mu > 0$ such that

$$\mathcal{B}(A_1, \mu) \subset \mathcal{B}(A_0, \lambda) \cap \mathcal{M}_{R,\varepsilon}. \tag{5.1}$$

Take a finite set A', $\chi(A_0, A') < \lambda/4$ and put $H' = \operatorname{span}(H \cup A')$. Apply Corollary 4.4 with respect to H', in the role of the space $\mathbb{E}^s$, and H as a subspace of H', to obtain $A_1 \in \mathcal{K}$, $\chi(A', A_1) < \lambda/4$, such that for every $h \in H$

$$\operatorname{card}(P(h, A_1) \cup Q(h, A_1)) \leq k + 2. \tag{5.2}$$

However, (5.2) remains true also in the space $\mathbb{E}$, and $\chi(A_0, A_1) < \lambda/2$.

Now fix γ, $0 < 2\gamma < \min\{\varepsilon, \operatorname{sep} A_1\}$. Either map associating with every $(h, X) \in H \times \mathcal{K}$ the sets $P(h, X)$ and $Q(h, X)$ is upper semi-continuous. Hence for each $h \in \mathsf{B}[h_0, R] \cap H$ there is $\delta(h) > 0$ such that if

$$X \in \mathcal{B}(A_1, \delta(h)) \quad \text{and} \quad y \in \mathsf{B}(h, \delta(h))$$

then

$$P(y, X) \subset P(h, A_1) + \mathsf{B}(\theta, \gamma), \quad Q(y, X) \subset Q(h, A_1) + \mathsf{B}(\theta, \gamma). \tag{5.3}$$

As $\mathsf{B}[h_0, R] \cap H$ is compact, it is covered by a finite collection of balls $\mathsf{B}(h_i, \delta(h_i))$, $i = 1, \dots, l$.

Take μ, $0 < \mu < \min\{\delta(h_1), \dots, \delta(h_l), \lambda/2\}$. With this choice of μ, (5.1) is fulfilled. In fact, let $X \in \mathcal{B}(A_1, \mu)$ and, for $h \in \mathsf{B}[h_0, R] \cap H$, let $W \subset P(h, X) \cup Q(h, X)$ be such that $\operatorname{card} W = k + 3$. Then for some j, $1 \leq j \leq l$, $h \in \mathsf{B}(h_j, \delta(h_j))$, and in view of (5.3),

$$W \subset (P(h, A_1) \cup Q(h, A_1)) + \mathsf{B}(\theta, \gamma). \tag{5.4}$$

However, due to (5.2) and the choice of γ, the set on the left side of (5.4) is contained in a union of a finite number of not more than $k + 2$ pairwise disjoint balls. Hence, there are at least two elements, say $x, y \in P(h, X) \cup Q(h, X)$, which belong to one and the same ball. Then $|x - y| < 2\gamma < \varepsilon$, which proves the claim as $\chi(A_0, A_1) < \lambda/2$ and $\mu < \lambda/2$. Now set

$$\mathcal{K}^{n,m} = \{X \in \mathcal{K}\colon C^{n,m}(X) \cap H \quad \text{is dense in } H\},$$
$$\hat{\mathcal{K}}^{n,m} = \{X \in \mathcal{K}\colon \hat{C}^{n,m}(X) \cap H \quad \text{is dense in } H\}.$$

In view of Proposition 3.1 and the claim, in order to complete the proof, it suffices to observe that

$$\hat{\mathcal{K}}^{n,m} \bigcap \Big(\bigcap_{R \in \mathbb{N}} \bigcap_{\varepsilon^{-1} \in \mathbb{N}} \mathcal{M}_{R,\varepsilon} \Big) \subset \mathcal{K}^{n,m}. \qquad \blacksquare$$

Remark 5.2: It is proved in the previous proposition that for typical $X \in \mathcal{K}$ the set $\hat{C}^{n,m}(X) \cap H$ is empty, provided $n + m \geq \dim H + 3$ and H is finite-dimensional.

The next theorem summarizes the main results in the paper:

THEOREM 5.3: *Suppose* $\mathbb{E}$, $\dim \mathbb{E} \geq 2$, *is either a finite-dimensional Euclidean space or a separable Hilbert space* $\mathbb{H}$, *and* H *is an affine subspace of* $\mathbb{E}$. *Then for every* $n, m \in \mathbb{N}$ *such that* $n + m - 2 \leq \dim H$, *and every typical compact set* $X \in \mathcal{K}$, *the set* $C^{n,m}(X) \cap H$ *is dense in* H. *Moreover, if* H *is of finite dimension,* $\dim H = k$ *and* $n + m > k + 2$, *then for a typical compact* $X \in \mathcal{K}$ *the upper common locus* $\hat{C}^{n,m}(X)$ *does not meet* H, *thus* X *induces a partition of* H, *by means of its common loci, into a finite sequence of dense sets* $C^{i,j}(X) \cap H$, $i + j = 2, \ldots, k + 2$.

Proof: Let $n, m \in \mathbb{N}$, $n + m - 2 \leq \dim H$, and let $(H_i)_{i=1}^{\infty}$ be a sequence of affine subspaces of H, all of dimension $n + m - 2$, such that $\bigcup_{i=1}^{\infty} H_i$ is dense in H. Simple observations show that Proposition 5.1 is true for the affine subspaces, thus for each H_i obtain a residual set $\mathcal{K}_i^{n,m}$ of compacta so that for every $X \in \mathcal{K}_i^{n,m}$ the set $C^{n,m}(X) \cap H_i$ is dense in H_i. Then $\mathcal{K}_H^{n,m} = \bigcap_{i=1}^{\infty} \mathcal{K}_i^{n,m}$ is residual in $\mathcal{K}$ and for every $X \in \mathcal{K}_H^{n,m}$ the set $C^{n,m}(X)$ is dense in H. Finally, put

$$\mathcal{K}_H = \bigcap \{\mathcal{K}_H^{n,m} \colon n, m \in \mathbb{N}, n + m - 2 \leq \dim H\}.$$

Every $X \in \mathcal{K}_H$ satisfies the conclusion of the theorem, the set $\mathcal{K}_H$ being residual in $\mathcal{K}$. For the second part of the theorem, refer also to Remark 5.2. ∎

Assigning in Theorem 5.3 $m = 1$ or $n = 1$ respectively, one obtains as corollaries results about common loci $L_P^n(X)$and $L_Q^m(X)$ of typical compacta $X \in \mathcal{K}$:

COROLLARY 5.4: *With* H *and* $\mathbb{E}$ *as in Theorem 5.3 and* $n \leq \dim H + 1$ *(resp.* $m \leq \dim H + 1$*), for a typical compact* $X \in \mathcal{K}$ *the set* $L_P^n(X) \cap H$ *(resp.* $L_Q^m(X) \cap H$*) is dense in* H. *If in addition* H *is finite-dimensional,* $\dim H = k$, *then also* $\hat{L}_P^{k+2}(X) \cap H$ *(resp.* $\hat{L}_Q^{k+2}(X) \cap H$*) is empty, hence a typical compact* $X \in \mathcal{K}$ *induces a partition of* H *into a finite sequence of dense disjoint sets:* $L_P^i(X) \cap H$, $i = 1, \ldots, k + 1$ *(resp.* $L_Q^j(X) \cap H$, $j = 1, \ldots, k + 1$*).*

Although the metric projection part of Corollary 5.4 is essentially contained in [BZ], this corollary brings new information for both finite-dimensional Euclidean spaces and infinite-dimensional separable Hilbert spaces.

A natural question to ask is about the validity of some of the above results in spaces more general than Hilbert spaces. One could expect this is the case, at least for separable uniformly convex Banach spaces. A hint to that might be the reference [BZh]. Also, it seems to us that common $(2, 2)$-loci of typical compacta should be dense in any separable strictly convex Banach space.

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